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On the max-domain of attractions of bivariate elliptical arrays

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Abstract Let $(U_{ni}, V_{ni}), 1 \le i \le n, n \ge 1$ be a triangular array of independent bivariate elliptical random vectors with the same distribution function as $\left(S_1, \rho_n S_1 + \sqrt{1-\rho_n^2} S_2\right), \rho_n \in (0,1)$ where (S_1, S_2) is a bivariate spherical random vector. Under assumptions on the speed of convergence of $\rho_n \to 1$ we show in this paper that the maxima of this triangular array is in the max-domain of attraction of a new max-id. distribution function $H_{\alpha,\lambda}$, provided that $\sqrt{S_1^2 + S_2^2}$ has distribution function in the max-domain of attraction of the Weibull distribution function Ψ_α.

Keywords Maxima of triangular arrays · Bivariate elliptical random vectors
Weibull max-domain of attraction · Hüsler-Reiss distribution · Coefficient of the upper tail dependence · Weak convergence

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1. Introduction

Let (S_1, S_2) be a bivariate spherical random vector with radius $R := \sqrt{S_1^2 + S_2^2}$ and let $(U_{ni}, V_{ni}), 1 \le i \le n, n \ge 1$ be a triangular array of independent bivariate elliptical random vectors with stochastic representation

$$\left(U_{ni}, V_{ni}\right) \stackrel{d}{=} \left(S_1, \rho_n S_1 + \sqrt{1-\rho_n^2} S_2\right), \quad 1 \le i \le n, n \ge 1, \tag{1}$$

where $\rho_n \in (-1,1)$ and $\stackrel{d}{=}$ means equality of distribution functions. Denote further by $M_{n1} := \max_{1 \le i \le n} U_{ni}, M_{n2} := \max_{1 \le i \le n} V_{ni}$ the componentwise maxima. Suppose

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that the distribution function F of R is in the max-domain of attraction of the unit Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in I\!R$, i.e., for constants $c(n) > 0, d(n)$

$$\lim_{n\to\infty} \sup_{x \in I\!R} \left| F^n(c(n)x + d(n)) - \Lambda(x) \right| = 0.$$

Let G denote the distribution function of S_1 with upper endpoint $\omega := \sup\{t : G(t) < 1\} = \sup\{t : F(t) < 1\}$ and put

$$b(n) := G^{-1}(1 - 1/n), \quad a(n) := \int_{b(n)}^{\omega} (1 - G(s))\, ds/(1 - G(b(n))), \quad n > 1.$$

Under the assumption

$$\lim_{n\to\infty} (1 - \rho_n) b(n)/a(n) = 2\lambda^2 \in [0, \infty] \tag{2}$$

Hashorva (2005a) shows the convergence in distribution

$$\big((M_{n1} - b(n))/a(n), (M_{n2} - b(n))/a(n)\big) \xrightarrow{d} (\mathcal{M}_1, \mathcal{M}_2), \quad n \to \infty \tag{3}$$

where the limiting random vector $(\mathcal{M}_1, \mathcal{M}_2)$ has distribution function given by

$$H_\lambda(x, y) = \exp\Big(-\exp(-x)\Phi\Big(\lambda + \frac{y - x}{2\lambda}\Big) - \exp(-y)\Phi\Big(\lambda + \frac{x - y}{2\lambda}\Big)\Big), \quad x, y \in I\!R,$$

with Φ the univariate standard Gaussian distribution.

Remarks:

(1) The bivariate Hüsler-Reiss distribution function H_λ (introduced in Hüsler and Reiss (1989)), is max-stable with unit Gumbel marginal distributions. For $\lambda = 0$ and $\lambda = \infty$ the asymptotic complete dependence and asymptotic independence of the components holds, respectively, in the limit i.e.,

$$H_0(x, y) = \min\big(\Lambda(x), \Lambda(y)\big), \quad x, y \in I\!R$$

and

$$H_\infty(x, y) = \Lambda(x)\Lambda(y), \quad x, y \in I\!R.$$

(2) The random vector (S_1, S_2) is spherically distributed if its distribution is invariant under orthogonal transformations in $I\!R^2$. See Cambanis et al. (1981), Fang et al. (1990) or Berman (1992).

Condition (2) for S_1, S_2 two independent standard Gaussian random variables was first imposed in Hüsler and Reiss (1989). In that context it is equivalent to

$$\lim_{n\to\infty} (1 - \rho_n) \ln n = \lambda^2 \in [0, \infty]. \tag{4}$$

Sibuya (1960) showed that if ρ_n does not depend on n then (3) holds with $\mathcal{M}_1, \mathcal{M}_2$ two independent unit Gumbel random variables. Condition (4) implies $\lim_{n\to\infty} \rho_n = 1$ which in its turn makes it possible to have a limiting distribution H_λ different from H_∞.

It is well-known from extreme value theory (for details consult one of the standard references: de Haan (1970), Leadbetter et al. (1983), Galambos (1987), Geluk and de Haan (1987), Resnick (1987), Reiss (1989), Berman (1992), Embrechts et al. (1997), Kotz and Nadarajah (2000), and Falk et al. (2004)) that F can be in the max-domain of attraction of either unit Gumbel distribution Λ, unit Weibull distribution $\Psi_\alpha(x) = \exp(-|x|^\alpha), \alpha > 0, x < 0$, or unit Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, $\alpha > 0, x > 0$.

If F is in the max-domain of attraction of Φ_α, then (see Berman (1992) and Eq. 12 below) the maxima of bivariate elliptical random sequences has asymptotic dependent components. So, for the Fréchet case letting $\rho_n \to 1$ is not of interest.

In this paper we investigate therefore only the case F is in the Weibull max-domain of attraction. If $\rho_n \to 1$, then it is possible to force the maxima to have asymptotic dependent components. More precisely, we prove in the present article that under the same condition on the sequence $\rho_n, n \geq 1$ as in the Gumbel case, the maxima of the bivariate elliptical triangular array above is in the max-domain of attraction of a new max-id. distribution function $H_{\alpha,\lambda}$ introduced in this paper (see below Eq. 9 and refer to Resnick (1987) for definition and properties of multivariate max-id. distribution functions).

There are several applications and softwares incorporating the max-stable Hüsler-Reiss distribution function H_λ. We expect that the new distribution function $H_{\alpha,\lambda}$ will extend the known applications for the case where the marginal distributions have finite upper endpoints and the distribution function of the random radius R is not in the Gumbel but in the Weibull max-domain of attraction.

2. Main result

In this section we provide first a well-known fact about the Weibull max-domain of attraction, needed to formulate the main result. From extreme value theory we know that the distribution function F is in the Weibull max-domain of attraction, i.e.,

$$\lim_{n\to\infty} \sup_{x<0} \left| F^n(c(n)x + d(n)) - \Psi_\alpha(x) \right| = 0 \tag{5}$$

if and only if $\omega < \infty$ and $1 - F(\omega - s) = s^{\alpha} L(s), s > 0$ with $L(s)$ a slowly varying function. Furthermore, the constants can be chosen as

$$c(n) := \omega - F^{-1}(1 - 1/n), \quad d(n) := \omega, \quad n > 1, \tag{6}$$

where F^{-1} is the generalised inverse of F. We assume for simplicity that the upper endpoint of F is 1. Next, let in the following ψ_α be a distribution function defined on $[-1, 1]$ by

$$\psi_\alpha(z) := \frac{\Gamma(\alpha + 3/2)}{\Gamma(\alpha + 1)\sqrt{\pi}} \int_{-1}^{z} (1 - s^2)^\alpha \, ds, \quad \alpha > 0, z \in [-1, 1].$$

Define ψ_α in the usual way in the whole real line putting $\psi_\alpha(z) := 0, \forall z \leq -1$ and $\psi_\alpha(z) := 1, \forall z \geq 1$.

We state now the main result of this paper:

Theorem 2.1: *Let (S_1, S_2) be a bivariate spherical random vector with almost surely positive random radius R with distribution function F. Let $(U_{ni}, V_{ni}), 1 \le i \le n, n \ge 1$ be a triangular array satisfying Eq. 1 with $\rho_n \in (0,1)$. Define the constants $a(n) := 1 - G^{-1}(1 - 1/n), n \in I\!N$ with G the distribution function of S_1 and G^{-1} its generalised inverse function. Assume that F has upper endpoint equal 1 and further F is in the max-domain of attraction of $\Psi_{\alpha^*}, \alpha^* > 0$ satisfying Eq. 5. If*

$$\lim_{n\to\infty} (1 - \rho_n)/a(n) = 2\lambda^2 \in [0, \infty] \tag{7}$$

holds, then

$$\lim_{n\to\infty} \sup_{(x,y)\in(-\infty,0)^2} \Big| \boldsymbol{P}\{M_{n1} \le 1 + a(n)x, M_{n2} \le 1 + a(n)y\} - H_{\alpha,\lambda}(x, y) \Big| = 0, \tag{8}$$

with

$$H_{\alpha,\lambda}(x, y) = \exp\Bigg(-|x|^\alpha \psi_\alpha\Bigg(\frac{1}{\sqrt{2|x|}} \Big[\lambda + \frac{y-x}{2\lambda} \Big] \Bigg) - |y|^\alpha \psi_\alpha\Bigg(\frac{1}{\sqrt{2|y|}} \Big[\lambda + \frac{x-y}{2\lambda} \Big] \Bigg) \Bigg), \quad x, y < 0 \tag{9}$$

where $\alpha := \alpha^* + 1/2$.

Remarks:

(1) Condition (7) is the same as the corresponding condition (2) for the Gumbel case since we can take $b(n) := 1, n \ge 1$. Further, it implies that $\lim_{n\to\infty} \rho_n = 1$ since $a(n) \to 0$ as $n \to \infty$.

(2) For the limiting cases $\lambda = 0$ and $\lambda = \infty$ the limiting distributions $H_{\alpha,0}$ and $H_{\alpha,\infty}$ have complete dependent/independent marginals, respectively.

(3) The bivariate distribution function $H_{\alpha,\lambda}$ is a max-id. distribution function. Its marginal distributions are unit Weibull with parameter α. See Resnick (1987) for properties of max-id. distribution functions.

(4) Elliptical random vectors are defined as linear transformation of spherical ones, i.e., an elliptical random vector $\boldsymbol{Z} \in I\!R^d, d \ge 2$ has stochastic representation $\boldsymbol{Z} \stackrel{d}{=} dAS + \boldsymbol{\mu}$, with A a d-dimensional square matrix, $\boldsymbol{S}$ a spherical random vector in $I\!R^d$ and $\boldsymbol{\mu} \in I\!R^d$. See Cambanis et al. (1981) or Fang et al. (1990) for more details.

(5) Let $(U_{n1}, V_{n1}), n \ge 1$ be bivariate elliptical random vectors with stochastic representation (1). In view of Lemma 3.3 of Hashorva (2005a) we have for any $n \in I\!N$

$$\boldsymbol{P}\{U_{n1} > u | V_{n1} > u\} = \frac{\int_{\arccos(\rho_n)/2}^{\pi/2} [1 - F(u/\cos(\theta))]\, d\theta}{\int_0^{\pi/2} [1 - F(u/\cos(\theta))]\, d\theta}, \quad \forall u > 0, \tag{10}$$

with F the distribution function of $\sqrt{S_1^2 + S_2^2}$. If $\lim_{n\to\infty} \rho_n = \rho \in [-1, 1)$ and F is in the Gumbel or Weibull max-domain of attraction we have [see Hashorva (2005b)]

$$\lim_{n\to\infty} \boldsymbol{P}\{U_{n1} > u_n | V_{n1} > u_n\} = 0 \tag{11}$$

for any $u_n \to \infty$ as $n \to \infty$. Equation 11 is the motivation for both Eqs. 2 and 7.

In case that $\lim_{n\to\infty}\rho_n = \rho \in (-1,1]$ and F is in the max-domain of attraction of $\Phi_\alpha, \alpha > 0$ we get using Eq. 10

$$\lim_{n\to\infty} \boldsymbol{P}\{U_{n1} > u_n \mid V_{n1} > u_n\} = \frac{\int_{\arccos(\rho)/2}^{\pi/2} \cos^\alpha(\theta)\, d\theta}{\int_0^{\pi/2} \cos^\alpha(\theta)\, d\theta} =: \lambda(\alpha,\rho) > 0. \tag{12}$$

The limit $\lambda(\alpha,\rho)$ is the so-called coefficient of the upper tail dependence for Z_1 and Z_2 with stochastic representation $(Z_1,Z_2) \overset{d}{=} (S_1, \rho S_1 + \sqrt{1-\rho^2}S_2)$. The fact that $\lambda(\alpha,\rho)$ is positive for any $\rho \in (-1,1]$ implies that the triangular array setup for F in the Fréchet max-domain of attraction is not interesting.

Hashorva (2005b) derives Eq. 12 in the case $\rho_n = \rho \in (-1,1], \forall n \geq 1$. See Frahm et al. (2003) for an alternative formula.

(6) Both the Hüsler-Reiss distribution H_λ and the bivariate distribution $H_{\alpha,\lambda}$ are exchangeable bivariate distributions. This is consequence of the fact that $(S_1,S_2) \overset{d}{=} (S_2,S_1)$ for any bivariate spherical random vector (S_1,S_2).

Next, we present a simple example.

Example 1: Let R be a positive random variable uniformly distributed in $(0,1)$ and let (O_1,O_2) be a bivariate random vector uniformly distributed on the unit circle of $\mathbb{R}^2$ being further independent of R. Define a random triangular array (U_{ni},V_{ni}), $1 \leq i \leq n, n \geq 1$ by the stochastic representation

$$(U_{ni},V_{ni}) \overset{d}{=} R(O_1, \rho_n O_1 + \sqrt{1-\rho_n^2}O_2), \quad 1 \leq i \leq n, n \geq 1,$$

with $\rho_n \in (-1,1), n \geq 1$. The random vector (U_{ni},V_{ni}) is elliptically distributed with $U_{ni} \overset{d}{=} V_{ni}, 1 \leq i \leq n, n \geq 1$. It is well-known that the uniform distribution on $(0,1)$ is in the max-domain of attraction of Ψ_1. With ρ_n, $a(n)$, and λ as in the above theorem we have that Eq. 8 holds with $\alpha = 3/2$. The new limiting distribution function is

$$H_{3/2,\lambda}(x,y) = \exp\left(-|x|^{3/2}\psi_{3/2}\left(\frac{1}{\sqrt{2|x|}}\left[\lambda + \frac{y-x}{2\lambda}\right]\right) - |y|^{3/2}\psi_{3/2}\left(\frac{1}{\sqrt{2|y|}}\left[\lambda + \frac{x-y}{2\lambda}\right]\right)\right),$$

$(x,y) \in (-\infty,0)^2$.

Straightforward calculations yield

$$\psi_{3/2}(z) = \frac{1}{\pi}\left[z(1-z^2)^{1/2}[2(1-z^2)/3+1] + \arcsin(z) + \pi/2\right], \quad \forall z \in [-1,1].$$

In order to prove the main theorem, we need a minor generalisation of Theorem 12.3.3 of Berman (1992).

Theorem 2.2: *Let W be a Beta distributed random variable with positive parameters a and b and let further $q_i : [0,\infty) \to [0,\infty), i = 1,2$ be two functions such that $q_2(u) > q_1(u), \forall u > 0$. Assume that the distribution function F with upper endpoint $\omega \in (0,\infty)$ is in the max-domain of attraction of $\Psi_\alpha, \alpha > 0$. If for $i = 1,2$*

$$\lim_{u\to\omega} \frac{q_i(u)}{2(\omega-u)} = q_i^* \tag{13}$$

holds, then we have

$$\begin{aligned}&\boldsymbol{E}\{\mathbf{1}(W \in [q_1(u), q_2(u)])[1-F(u(1-W)^{-1/2})]\}\\ &\quad = (1+o(1))(\omega-u)^a 2^a [1-F(u)] \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_{\min(q_1^*,1)}^{\min(q_2^*,1)} (1-s)^{\alpha} s^{a-1}\, ds, \quad u \to \omega, \quad (14)\end{aligned}$$

with $1(\cdot \in [q_1(u), q_2(u)])$ *the indicator function.*

Proof of Theorem 2.2: If $q_1(u)$ equals 0 and $q_2(u)$ equals 1 for all $u > 0$, then the claim is shown in Theorem 12.3.3 of Berman (1992). The other case follows along the same lines of the proof of the aforementioned theorem. ■

Proof of Theorem 2.1: By the assumptions $S_1, S_2, U_{ni}, V_{ni}, 1 \le i \le n, n \ge 1$ have common distribution function G with upper endpoint 1. Further by Corollary 12.1.1 of Berman (1992) we have for all $x > 0$

$$1 - G(x) = \boldsymbol{E}\{1 - F(x(1-W)^{-1/2}\}/2, \tag{15}$$

with W a Beta distributed random variable with parameters 1/2,1/2. Hence in view of Theorem 12.3.3 of Berman (1992) or directly by Eq. 14 we obtain for all $s \le 0$

$$\lim_{n\to\infty} \boldsymbol{P}\{M_{n1} \le t_n(s)\} = \lim_{n\to\infty} \boldsymbol{P}\{M_{n2} \le t_n(s)\} = exp(-\mid s\mid^{\alpha}) \tag{16}$$

where $\alpha := \alpha^* + 1/2, t_n(s) := 1 + a(n)s$, with $a(n) := 1 - G^{-1}(1-1/n), n > 1$. In view of Lemma 4.1.3 of Falk et al. (2004) in order to complete the proof we need to determine further the following function

$$L(x,y) := \lim_{n\to\infty} n\boldsymbol{P}\{U_{n1} > t_n(x), V_{n1} > t_n(y)\}, \quad x, y < 0.$$

Note in passing that the above limit need not exist in general. By the assumptions we have the stochastic representation

$$(U_{n1}, V_{n1}) \stackrel{d}{=} \left(S_1, \rho_n S_1 + \sqrt{1-\rho_n^2} S_2\right) \stackrel{d}{=} \Big(R\, cos(\Theta), R\, cos(\Theta - \tau_n)\Big), \quad n \ge 1,$$

with Θ a random angle uniformly distributed on $(-\pi, \pi)$ and $\tau_n := \arccos(\rho_n)$. Hence for $x > 0, y \ge 0$ we get (see also Lemma 3.3 of Hashorva (2005a))

$$\begin{aligned}&\boldsymbol{P}\{U_{n1} > x, V_{n1} > y\}\\ &\quad = \frac{1}{2\pi}\int_{\min(\beta_n(x,y),\pi/2)}^{\pi/2} [1-F(x/\cos(\alpha))]\, d\alpha + \frac{1}{2\pi}\int_{\tau_n-\pi/2}^{\max(\beta_n(x,y),\tau_n-\pi/2)} [1-F(y/\cos(\alpha-\tau_n))]\, d\alpha\\ &\quad = \frac{1}{2\pi}\int_{\min(\beta_n(x,y),\pi/2)}^{\pi/2} [1-F(x/\cos(\alpha))]\, d\alpha + \frac{1}{2\pi}\int_{-\pi/2}^{\max(\beta_n(x,y)-\tau_n,-\pi/2)} [1-F(y/\cos(\alpha))]\, d\alpha\\ &\quad = \frac{1}{2\pi}\int_{\min(\beta_n(x,y),\pi/2)}^{\pi/2} [1-F(x/\cos(\alpha))]\, d\alpha + \frac{1}{2\pi}\int_{\min(\widetilde{\beta}_n(x,y),\pi/2)}^{\pi/2} [1-F(y/\cos(\alpha))]\, d\alpha,\end{aligned}$$

with $\beta_n(x,y) := \arctan((y/x - \rho_n)/\sin(\tau_n))$ and $\widetilde{\beta}_n(x,y) := \tau_n - \beta_n(x,y), n \ge 1$.

For any x, y negative we have

$$\lim_{n\to\infty} t_n(x) = \lim_{n\to\infty} t_n(y) = 1,$$

hence we may write for large n

$$\begin{aligned}\boldsymbol{P}\{U_{n1} > t_n(x), V_{n1} > t_n(y)\} &= \frac{1}{2\pi}\int_{\min(\beta_n(t_n(x),t_n(y)),\pi/2)}^{\pi/2} [1 - F(t_n(x)/\cos(\alpha))]\, d\alpha \\ &\quad + \frac{1}{2\pi}\int_{\min(\widetilde{\beta}_n(t_n(x),t_n(y)),\pi/2)}^{\pi/2} [1 - F(t_n(y)/\cos(\alpha))]\, d\alpha \\ &=: I_n(x, y) + J_n(x, y).\end{aligned}$$

Condition (7) implies $\rho_n \to 1$ and $\tau_n/\sqrt{2(1-\rho_n)} = 1 + O(1-\rho_n)$ for $n \to \infty$. We calculate next the limit of both integrals $I_n(x, y), J_n(x, y)$ above.

We consider only the case $\lambda \in (0, \infty)$. The other cases ($\lambda = 0$ and $\lambda = \infty$) follow using similar arguments. Assume that $\liminf_{n\to\infty} \beta_n(t_n(x), t_n(y)) \geq 0$. Changing the variables we get for large n

$$I_n(x, y) = \boldsymbol{E}\{\mathbf{1}(W \in [\min(1, \sin^2(\beta_n(t_n(x), t_n(y)))), 1])[1 - F(t_n(x)(1 - W)^{-1/2})]\}/4.$$

By the assumptions we may write

$$\begin{aligned}\lim_{n\to\infty} \frac{\beta_n(t_n(x), t_n(y))}{\sqrt{a(n)}} &= \lim_{n\to\infty} \frac{\arctan((t_n(y)/t_n(x) - \rho_n)/\sin(\tau_n))}{\sqrt{a(n)}} \\ &= \lim_{n\to\infty} \frac{1 - \rho_n + t_n(y)/t_n(x) - 1}{\sqrt{2a(n)(1-\rho_n)}} = \lambda + \frac{y - x}{2\lambda}.\end{aligned}$$

Hence Eqs. 15, 16 and Theorem 2.2 yield

$$\begin{aligned}\lim_{n\to\infty} nI_n(x, y) &= \lim_{n\to\infty} n[1 - G(t_n(x))] \lim_{n\to\infty}\left(\frac{nI_n(x, y)}{n[1 - G(t_n(x))]}\right) \\ &= \lim_{n\to\infty} \frac{n[1 - G(t_n(x))]}{2c_\alpha} \int_{\min(\lim_{n\to\infty}\sin^2(\beta_n(t_n(x),t_n(y)))/(2a(n)x),1)}^{1} (1 - s)^\alpha s^{-1/2}\, ds \\ &= \frac{|\, x\,|^\alpha}{2c_\alpha} \int_{\min((\lambda+(y-x)/(2\lambda))^2/(2|x|),1)}^{1} (1 - s)^\alpha s^{-1/2}\, ds \\ &= |\, x\,|^\alpha \left[1 - \psi_\alpha\left(\frac{1}{\sqrt{2|x|}}\left[\lambda + \frac{y - x}{2\lambda}\right]\right)\right],\end{aligned}$$

with

$$c_\alpha := \int_0^1 (1 - s)^\alpha s^{-1/2}\, ds = \frac{\Gamma(\alpha + 1)\sqrt{\pi}}{\Gamma(\alpha + 3/2)}.$$

If $\liminf_{n\to\infty}\beta_n(t_n(x), t_n(y)) < 0$ we write for large n

$$\begin{aligned} I_n(x,y) &= \boldsymbol{E}\{\mathbf{1}(W \in [0,1])[1 - F(t_n(x)(1-W)^{-1/2})]\}/4 \\ &\quad + \boldsymbol{E}\{\mathbf{1}(W \in [0, sin^2(\beta_n(t_n(x), t_n(y)))])[1 - F(t_n(x)(1-W)^{-1/2})]\}/4. \end{aligned}$$

Hence this case can be handled similarly using further the result of Theorem 2.2. By the assumptions

$$\lim_{n\to\infty} \frac{\widetilde{\beta}_n(t_n(x), t_n(y))}{\sqrt{a(n)}} = \lim_{n\to\infty} \frac{\tau_n - \beta_n(t_n(x), t_n(y))}{\sqrt{a(n)}} = \lambda + \frac{x-y}{2\lambda}$$

consequently using the same arguments as above we obtain

$$\lim_{n\to\infty} J_n(x,y) = |y|^\alpha \left[1 - \psi_\alpha\left(\frac{1}{\sqrt{2|y|}}\left[\lambda + \frac{x-y}{2\lambda}\right]\right)\right],$$

thus the function $L(x,y)$ is given by

$$\begin{aligned} &|x|^\alpha \left[1 - \psi_\alpha\left(\frac{1}{\sqrt{2|x|}}\left[\lambda + \frac{y-x}{2\lambda}\right]\right)\right] + |y|^\alpha \left[1 - \psi_\alpha\left(\frac{1}{\sqrt{2|y|}}\left[\lambda + \frac{x-y}{2\lambda}\right]\right)\right], \\ &\quad \forall (x,y) \in (-\infty, 0)^2, \end{aligned}$$

hence the proof follows. ■

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